

# Limits of Local Thermodynamic Equilibrium in Short Spaced Cesium Plasmas

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Limits of validity of the assumption of local thermodynamic equilibrium in short spaced cesium plasmas are deduced from necessary conditions for the existence of local translational, internal, and radiation equilibrium. Results are presented graphically by curves of total pressure or electron density as a function of electron temperature required to satisfy the given conditions.

## Introduction

The exact definition of local thermodynamic equilibrium (LTE) is not standard. In rather broad terminology, one may say that LTE exists in a given plasma volume if the effects of deviations from thermodynamic equilibrium in that volume are negligible. The state of equilibrium is characterized by Maxwellian distributions for the heavy particles and the free electrons, a Saha-Boltzmann distribution for the bound electronic states, and a Planckian distribution for the radiation field.

The question of the existence of LTE in plasmas has been studied by previous authors<sup>1–5</sup>. Most of the theoretical investigations attempt to establish criteria involving electron temperatures and densities which guarantee Saha-Boltzmann equilibrium of the bound states in an optically thin plasma containing thermalized electrons and hydrogenic atoms and ions. For practical reasons, most of the experimental investigations have been conducted with noble gas discharges. An excellent review of particularly this aspect of the work has been given recently by Richter<sup>4</sup>. On the other hand, technological interest in thermionic diodes has spawned considerable effort directed toward an understanding of cesium plasmas. Here the work of Norcross<sup>5</sup> is central. His interest is, however, primarily in obtaining theoretical predictions of experimentally determined temperature and density profiles in a diode, rather than in treating the general question of the existence of LTE. In this paper, we consider conditions required for establishing LTE in short spaced cesium plasmas with total pressures in the range 0.01 Torr to 100 Torr and electron temperatures in the range 2000 K to 3000 K. In addition to an investigation of conditions necessary for Saha-Boltzmann equilibrium,

we also considered LTE in translational degrees of freedom and in the radiation field. The ranges of parameters chosen reflects our interest in thermionic diodes operating in the arc-mode. However, the results of our investigations are of general applicability.

In this study, we have neglected all molecular effects and have assumed the ions to be present only in the ground state. The only inelastic processes considered are those involving electrons. Except for the calculation of transmission probabilities, the doublets were coalesced to form one level of energy

$$E = [(2J_1 + 1)E_1 + (2J_2 + 1)E_2] / 2(2L + 1)$$

where  $E_{1,2}$  are the doublet energies,  $J_{1,2}$  are the total angular momentum quantum numbers, and  $L$  is the orbital angular momentum quantum number.

## Conditions for LTE

We consider the principal conditions necessary for the establishment of LTE in a localized plasma volume to be:

1. For the distributions to be Maxwellian, the particles must be collision dominated in the sense that energy relaxation lengths are much smaller than characteristic dimensions of the volume. Also inelastic and superelastic collisions should balance for low electron densities.

2. In order that Saha-Boltzmann equilibrium exist at the *electron temperature*, the loss of excited atoms due to radiation processes and excited state diffusion out of the volume must be negligibly small compared to collisionally induced transitions. The electron distribution function must, of course, also be Maxwellian.

3. For Planckian equilibrium of the radiation field, it is necessary that the plasma be optically thick to the emitted spectrum.

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### Evaluation of LTE Conditions

For our numerical calculations, we considered a heavy particle temperature of 1000 °K and spacings of 0.25 mm and 5.0 mm. This does not constitute a simplification, as the same calculations may be carried out for any values of these parameters. The spacings represent extreme conditions encountered in diode application and the heavy particle temperature is characteristic of diode plasmas. For brevity we present only typical results. More complete results may be found in the Reference 6.

#### 1. Collision Dominance of the Electrons

The dominant processes in a plasma are governed by the electrons. We, therefore, have considered only the relaxation lengths of the electrons in relation to the question of collision dominance.

Because of our interest in short spaced plasmas, we found it necessary to consider relaxation lengths as functions of energy rather than simple mean free paths. That is, because of their energy dependence, the characteristic relaxation lengths corresponding to separate parts of an arbitrary distribution function may differ sufficiently to necessitate, in the case of a short spaced plasma, separate analytic treatments (see Discussion). In this study, we ignored all collective effects, treating the electrons as test particles entering a plasma in which the background distributions were assumed to be Maxwellian.

In the diffusion approximation, the energy relaxation length for a monoenergetic beam of test electrons is given by <sup>6</sup>

$$\lambda_- = \sqrt{\frac{2}{3}} \lambda_{-\sigma}^P \lambda_-^E. \quad (1)$$

Here  $\lambda_{-\sigma}^P$  is the collisional relaxation length for momentum transfer between test electrons and heavy particles and  $\lambda_-^E$  is the total collisional relaxation length for energy transfer. For a multispecies background, total collisional relaxation lengths are defined by

$$(\lambda_{\alpha}^{P,E})^{-1} = \sum_{\beta} (\lambda_{\alpha\beta}^{P,E})^{-1} \quad (2)$$

where  $\lambda_{\alpha\beta}^{P,E}$  are the collisional relaxation lengths for test particles of species  $\alpha$  colliding with background particles of species  $\beta$ . Employing a Boltzmann collision operator with a cutoff,  $(\partial f_{\alpha}/\partial t)_{\text{coll}}$ ,  $\lambda_{\alpha\beta}^{P,E}$  are obtained from

$$\int d\mathbf{v} m_{\alpha} \mathbf{v} (\partial f_{\alpha}/\partial t)_{\text{coll}} = - \frac{n_{\alpha} |V_{\alpha}|}{\lambda_{\alpha\beta}^P(\epsilon_{\alpha})} m_{\alpha} \mathbf{V}_{\alpha} \quad (3)$$

and

$$\left| \int d\mathbf{v} \frac{1}{2} m_{\alpha} v^2 (\partial f_{\alpha}/\partial t)_{\text{coll}} \right| = \frac{n_{\alpha} |V_{\alpha}|}{\lambda_{\alpha\beta}^E(\epsilon_{\alpha})} \frac{1}{2} m_{\alpha} V_{\alpha}^2 \quad (4)$$

with

$$f_{\alpha} = n_{\alpha} \delta(\mathbf{v} - \mathbf{V}_{\alpha}), \quad (5)$$

$$f_{\beta} = n_{\beta} (2\pi/m_{\beta} k T_{\beta})^{3/2} \exp\{-m_{\beta} v^2/2 k T_{\beta}\}. \quad (6)$$

Here  $m_{\alpha}, \beta$ , and  $n_{\alpha}, \beta$  are the masses and average densities of species  $\alpha$  and  $\beta$ ,  $T_{\beta}$  is the temperature of the background particles, and  $\epsilon_{\alpha}$  ( $= \frac{1}{2} m_{\alpha} V_{\alpha}^2$ ) is the energy of the test particle.

In using Eq. (1) as the characteristic length for energy transfer (Maxwellization length), it is presupposed that  $\lambda_-^P \ll \lambda_-^E$ .

In the interest of application, we present in Fig. 1 ranges of total pressure and electron temperature for which Maxwellization is sufficient for given electron energies. The electron-neutral cross sections given by Nighan <sup>7</sup> were used in these calculations. For high energies ( $\epsilon_- = 2.0$  eV), we have indicated the effect of inelastic collisions exciting the first resonance level ( $\lambda_-^E = \lambda_{-,0 \rightarrow 1}^E$ ).

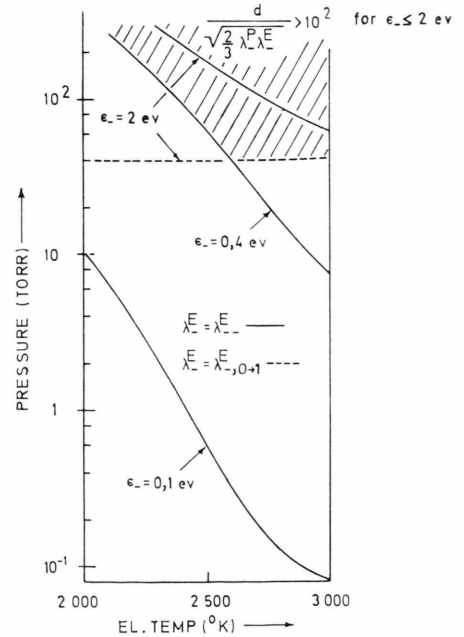


Fig. 1. Curves of total pressure as a function of electron temperature satisfying the requirement of sufficient Maxwellization, in the diffusion approximation, i.e.  $d / \sqrt{\frac{2}{3}} \lambda_{-\sigma}^P \lambda_-^E = 10^2$ , for given test electron energies. A plasma spacing of  $d = 0.25$  mm is assumed. Shaded region is that for sufficient Maxwellization for electron energies  $\epsilon_- \leq 2.0$  eV.

## 2. Saha-Boltzmann Equilibrium of Excited States

The ratio of the collisionally induced transition rate between levels  $i$  and  $j$  to the radiative transition rate from level  $i$  is

$$R_{i \rightarrow j} = N_e K_{i \rightarrow j} / \sum_{k < i} A_{i \rightarrow k} \quad (7)$$

where  $A_{i \rightarrow k}$  is the Einstein spontaneous emission coefficient,  $K_{i \rightarrow j}$  is the collisional transition rate coefficient, and  $N_e$  is the electron density.

Saha-Boltzmann equilibrium may exist among certain groups of levels, while it does not exist among all bound states. Since the level spacing diminishes with increasing (principle) quantum number, and since the collisional transition rate between bound states increases with decreasing energy spacing, such a group may always be found among the high quantum number states<sup>8</sup>. For cesium the relative spacing

of neighboring P and D states is also very small even for low quantum numbers. Hence, Saha-Boltzmann equilibrium will exist between these states under conditions insufficient to insure its existence over the entire level scheme.

In Fig. 2, we present ranges of electron densities and temperatures necessary to insure Saha-Boltzmann equilibrium between the low-lying P and D states based on the requirement  $R_{i \rightarrow j} \geq 10$ .

In these calculations, collisional transition rates were obtained from classical cross sections derived by Gryzinski<sup>9</sup>, and from the distorted wave approximation<sup>10</sup> assuming a Maxwellian electron background with an average density given by the Saha equation and using an effective Gaunt factor

$$\bar{g} = (3^{3/2}/2\pi) 0.15 \{ [\varepsilon_- / (\varepsilon_i - \varepsilon_j)] - 1 \}$$

for transitions between states  $i$  and  $j$ . The spontaneous emission coefficients obtained by Norcross and Stone<sup>8</sup> were used and the plasma was assumed to be optically thin.

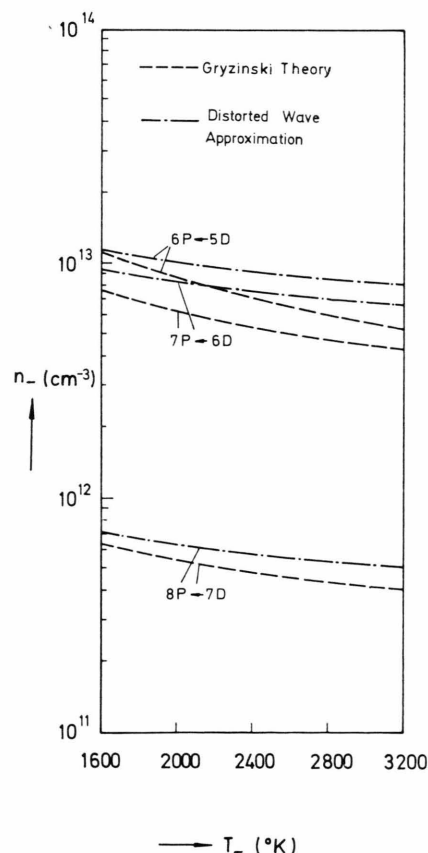


Fig. 2. Curves of electron density,  $n_e$ , as a function of electron temperature,  $T_e$ , satisfying the requirement that transitions between low-lying P and D states are collision dominated, i.e.  $R_{i \rightarrow j} = 10$  (see Eq. (7)), and are hence in relative Saha-Boltzmann equilibrium with the free electrons.

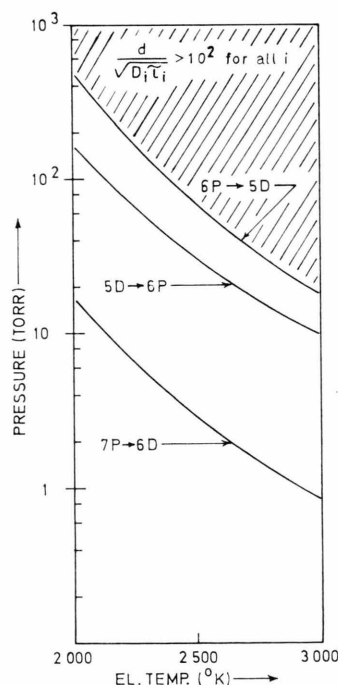


Fig. 3. Curves of total pressure as a function of electron temperature satisfying the requirement that diffusion of the excited state  $i$  is negligible, i.e.  $d / \sqrt{D_i} \tau_i = 10^2$ , for the three lowest excited states  $i = 6P, 5D, 7P$ . A plasma spacing of  $d = 0.25$  mm is assumed. Shaded region is that for negligible diffusion for all excited states.

The characteristic length for the diffusion of an excited state,  $i$ , out of the plasma volume is

$$l_i^{\text{DIFF}} = \sqrt{D_i \tau_i} \quad (8)$$

where  $D_i$  is the diffusion coefficient for the state  $i$ , assumed independent of  $i$ , and  $\tau_i$  is the relaxation time related to the fastest collisional process altering the state  $i$ .

In Fig. 3, we present ranges of total pressure and electron temperature for which excited state diffusion is negligible. The collisional transition rates were obtained from the distorted wave approximation with an average electron density given by the Saha equation. The shaded region is that for which diffusion is unimportant for all levels.

### 3. Equilibrium of the Radiation Field

The optical properties of the plasma are described in terms of the transmission probabilities,  $T_{i \rightarrow j}$ , for radiation corresponding to the transition  $i \rightarrow j < i$ . The transmission probability for radiation of frequency  $\nu_{ij}$  characteristic of the transition  $i \rightarrow j$ , in a

homogeneous plasma of thickness  $l$  is

$$T_{i \rightarrow j}(l) = \int_{-\infty}^{+\infty} d\nu b(\nu - \nu_{ij}) \exp\{-b(\nu - \nu_{ij}) \kappa l\}. \quad (9)$$

Here  $b(\nu - \nu_{ij})$  is the line shape factor and

$$\kappa = (n_j e^2 / 4 \epsilon_0 m_e c) f_{j \rightarrow i}$$

where  $e$  and  $f_{j \rightarrow i}$  are respectively the electronic charge in Coulombs and the absorption oscillator strength for the transition  $j \rightarrow i$ . Transmission probabilities near unity correspond to an optically thin plasma while those near zero correspond to an optically thick plasma.

In Fig. 4, we present ranges of total pressure and electron temperature for which optically thick or thin conditions prevail. We calculated transmission probabilities for transitions from the levels  $n P_{1/2}$  to the ground state ( $6S_{1/2}$ ) considering separately Doppler and pressure (resonance and Stark) broadening. The absorption oscillator strengths for cesium were obtained from the work of Stone<sup>11</sup>.

### Discussion

The results presented in Fig. 1 indicate that there are regions of technological interest for which the electrons are not sufficiently Maxwellized. In these regions it is pertinent to investigate the test particle relaxation process in detail. For this purpose, we calculated the collisional relaxation lengths defined by Eqs. (3) and (4) as functions of test electron energy for given total pressures and electron temperatures. The results of these calculations, along with the corresponding diffusion lengths from Eq. (1), are presented in Figure 5.

Based on the relative values of these relaxation lengths and their relations to the plasma spacing, it is apparent that the electrons must be subdivided into three groups each requiring separate analytical treatment. These groups correspond to those electrons which are i) randomized and Maxwellized, ii) randomized but insufficiently Maxwellized, iii) neither sufficiently randomized nor sufficiently Maxwellized. The first group normally contains low energy electrons, although at low temperatures (2000 °K) and high pressures (1 Torr to 10 Torr), electrons with energies in the neighborhood of 1.0 eV may also be included. This group is sufficiently well described by a set of transport equations obtained

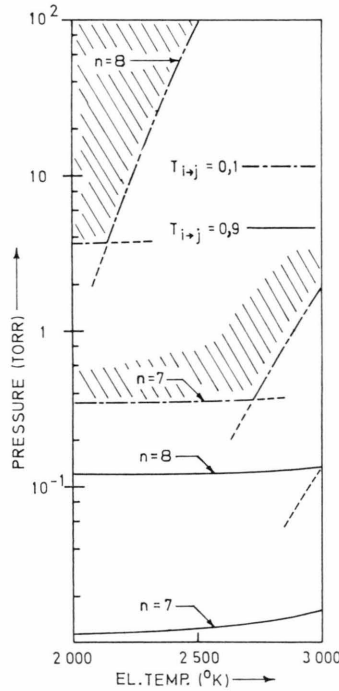


Fig. 4. Curves of total pressure as a function of electron temperature yielding transmission probabilities  $T_{i \rightarrow j} = 0.1; 0.9$  for two members of the principal line series  $n P_{1/2} \rightarrow 6 S_{1/2}$  ( $n = 7, 8$ ). Horizontal curves are for Doppler broadening. Diagonal curves are for pressure broadening. A plasma spacing  $d = 0.25$  mm is assumed. Shaded region is that for which  $T_{i \rightarrow j} < 0.1$ .

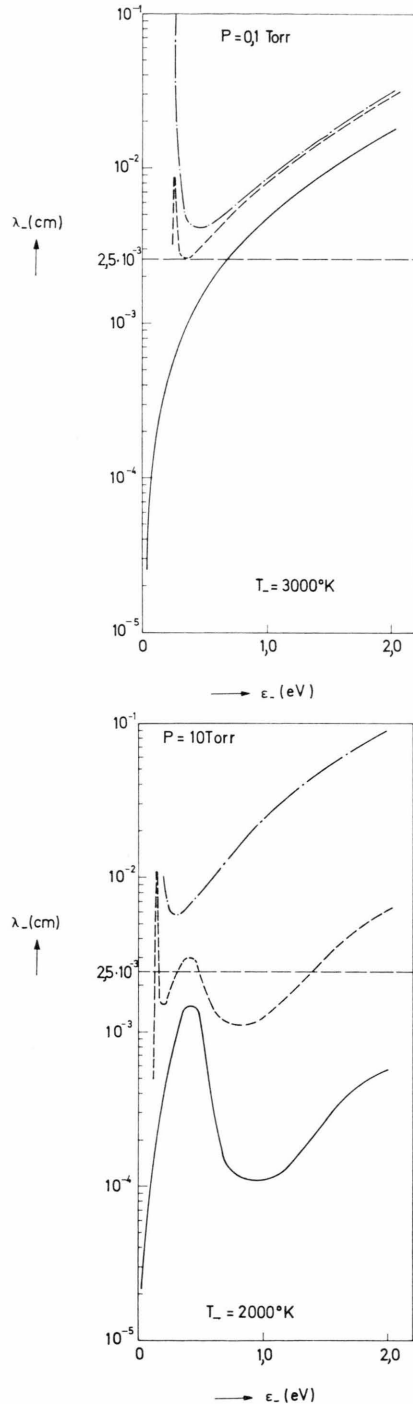


Fig. 5. Electron relaxation lengths as functions of test electron energy,  $\epsilon_-$ . Legend: Collisional energy relaxation length for collisions with background electrons,  $\lambda_{-}^E$  [see Eq. (4)] = ———; Energy relaxation length in diffusion approximation,  $\lambda_{-} = 1/3 \lambda_{-}^E$  [see Eq. (1)] = - - - - -; Collisional momentum relaxation length,  $\lambda_{-}^P$  [see Eqs. (2) and (3)] = — · — · —. Horizontal dashed line represents the condition  $\lambda_{-} = d/10$ ,  $d = 0.25$  mm.

via a Chapman-Enskog procedure. The second group contains electrons of moderate energies and may also include those with energies in the neighborhood of the Ramsauer minimum (0.4 eV). The treatment of these electrons must be based on a kinetic equation which may, because of sufficient randomization, be solved by expansion in spherical harmonics. The third group, containing high energy electrons, also requires a detailed treatment involving a kinetic equation. A development in spherical harmonics is, however, no longer possible due to insufficient randomization. Because of their role in inelastic processes, an accurate treatment of these high energy electrons is in any case necessary.

The results presented in Fig. 2 may be compared with those of Norcross and Stone<sup>8</sup> obtained under the same assumptions, i. e. that the plasma is optically thin to the lines considered. Figure 2 indicates that at an electron temperature of 2000°K Saha-Boltzmann equilibrium will exist between the levels 6P and 5D and between 7P and 6D at an electron density of  $10^{13} \text{ cm}^{-3}$  and between the levels 8P and 7D at  $10^{12} \text{ cm}^{-3}$ . The results of Norcross and Stone are given in terms of reduced densities,  $\rho(p)$ , defined as the ratio of the actual density,  $N(p)$ , of the state  $p$  to the corresponding Saha density. If  $\rho(l) = \rho(m)$  for two states  $l$  and  $m$ , then they will be in relative Saha-Boltzmann equilibrium. That is

$$N(l)/N(m) = (g_l/g_m) \exp\{-(E_m - E_l)/kT_e\}$$

where  $g_{l,m}$  and  $E_{l,m}$  are the corresponding statistical weights and ionization energies. From Fig. 3 of Ref. 7, we see that an electron temperature of 2000°K and density of  $10^{13} \text{ cm}^{-3}$ ,  $\rho(5D)/\rho(6P) = 0.87$  and  $\rho(6D)/\rho(7P) = 1.0$  while at  $10^{12} \text{ cm}^{-3}$ ,  $\rho(7D)/\rho(8P) = 1.0$ , in substantial agreement with our predictions.

The results of Fig. 2 are strictly valid only when the diffusion of excited states out of the volume is negligible. As indicated in Fig. 3, however, diffusion is important over a rather large range of electron temperatures and total pressures for plasma spacings  $\sim 0.25$  mm. The effect of diffusion will be to raise the electron densities required to guarantee Saha-Boltzmann equilibrium. The densities given in Fig. 2 must, therefore, be considered minimal provided the plasma is optically thin.

The results presented in Fig. 4 indicate that while an optically thick assumption is approximately valid



for radiation arising from transitions from the first resonance level at total pressures above  $10^{-1}$  Torr and  $10^{-2}$  Torr and spacings of 0.25 mm and 5.0 mm respectively, it is invalid for transitions from higher resonance levels over a wide range of pressures. This optically grey condition should be considered in a realistic treatment.

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## Turbulent Diffusion \* I.

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The problem of turbulent diffusion is re-examined. Its close connection with Schwinger's formulation of quantum field theory is established. For the transition probabilities we find a Dyson equation by using Zwanzig formalism with renormalization. The first nontrivial approximation of the memory kernel yields Saffman's result.

### § 1. The Problem

The phenomenon of turbulence was discovered by Reynolds using ink as an indicator. For a given turbulent field one may therefore consider the corresponding spread out of the ink, i. e. we require the transition probability.

If the test particle is very small, one may expect it to follow immediately the motion of the surrounding fluid, according to <sup>1</sup>

$$\dot{x} = v(t, x). \quad (1)$$

For a finite size of the particle, we expect some kind of relaxation which has been carefully estimated by Hinze <sup>2</sup>. If  $v$  were independent of  $x$ , we should have Brownian motion. This may serve us later on as a reference <sup>3</sup>.

Starting from a fixed point, say  $x = x_0$ ,  $x(t)$  becomes a stochastic process in the course of time and hence the right-hand side of Eq. (1) develops into a

stochastic function of another one, which is difficult to handle. In principle, the problem is as follows: Let the statistics of  $v(t, x)$  be given by the sequence of its joint-probability distributions <sup>4</sup>:

$$\begin{aligned} P_1(t_1, x_1 : v_1) dv_1 \\ P_2(t_1, x_1 : v_1; t_2, x_2 : v_2) dv_1 dv_2 \\ \vdots \end{aligned} \quad (2)$$

For any realization  $\omega$  of the turbulent fluid we have

$$x(x_0, t, \omega) = x_0 + \int_0^t v(t', x(t', \omega), \omega) dt'. \quad (3)$$

For different realizations we get a distribution function  $p(t, x_0 : x)$  of the positions at time  $t$  which we may approximate as follows: Let us assume that the paths are rather continuous and can be approximated by a polygon. Then for  $t = n \tau$ , we have:

$$\begin{aligned} p(t, x_0 : x) \approx \int \cdots \int P_{n+1}[0, x_0 : (x_1 - x_0)/\tau; \\ \tau; x_1 : (x_2 - x_1)/\tau; \dots; t, x_n : (x - x_{n-1})/\tau] \\ dx_1 dx_2 \dots dx_n \end{aligned} \quad (4)$$

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